ρ-calculus modulo

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Definition

Terms :

$$
\begin{array}{rll}
T ::= & & \\
& | x & \text{Variable} \\
& | a & \text{Constant} \\
& | T.T & \text{Application} \\
& | T \to T & \text{Abstraction} \\
& | T \mid T & \text{Structure (non-determinism)}
\end{array}
$$

Definition

The classical rules of the ρ-calculus are :

$$(\rho) \quad (M_1 \to M_2).M_3 \quad \mapsto \quad \sigma_1 M_2 \mid \ldots \mid \sigma_n M_2 \quad \text{with } \{\sigma_i\} = \mathcal{S}ol(M_1 \ll M_3)$$

$$(\delta) \quad (M_1 \mid M_2).M_3 \quad \mapsto \quad M_1.M_3 \mid M_2.M_3$$

Theories

Theories can be plugged inside the calculus through the matching definition :

$$\mathcal{S}ol(P \ll M) = \{\sigma / \sigma P =_T M\}$$

where T can be any theory.

About the patterns

- Is it safe to use terms such as $(x \mid y \longrightarrow x)$?
  What is the meaning of $(x \mid y \longrightarrow x).(a \mid (a \mid b))$ anyway ?
  Is $\mid$ ACI ? AC ? A ? syntactic ?

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  Is $\mid$ ACI ? AC ? A ? syntactic ?
- Should we introduce higher order matching ?

About the patterns

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The non-linear terms problem

$$(x \to f(x,x)).(a \mid b)$$

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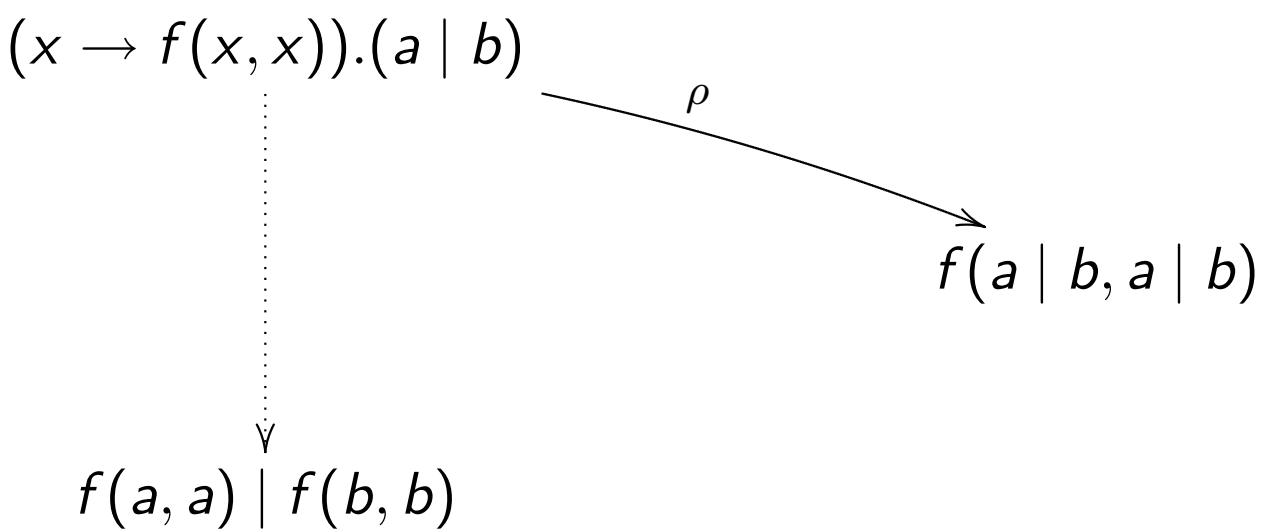
$$f(a,a) \mid f(b,b)$$

The non-linear terms problem



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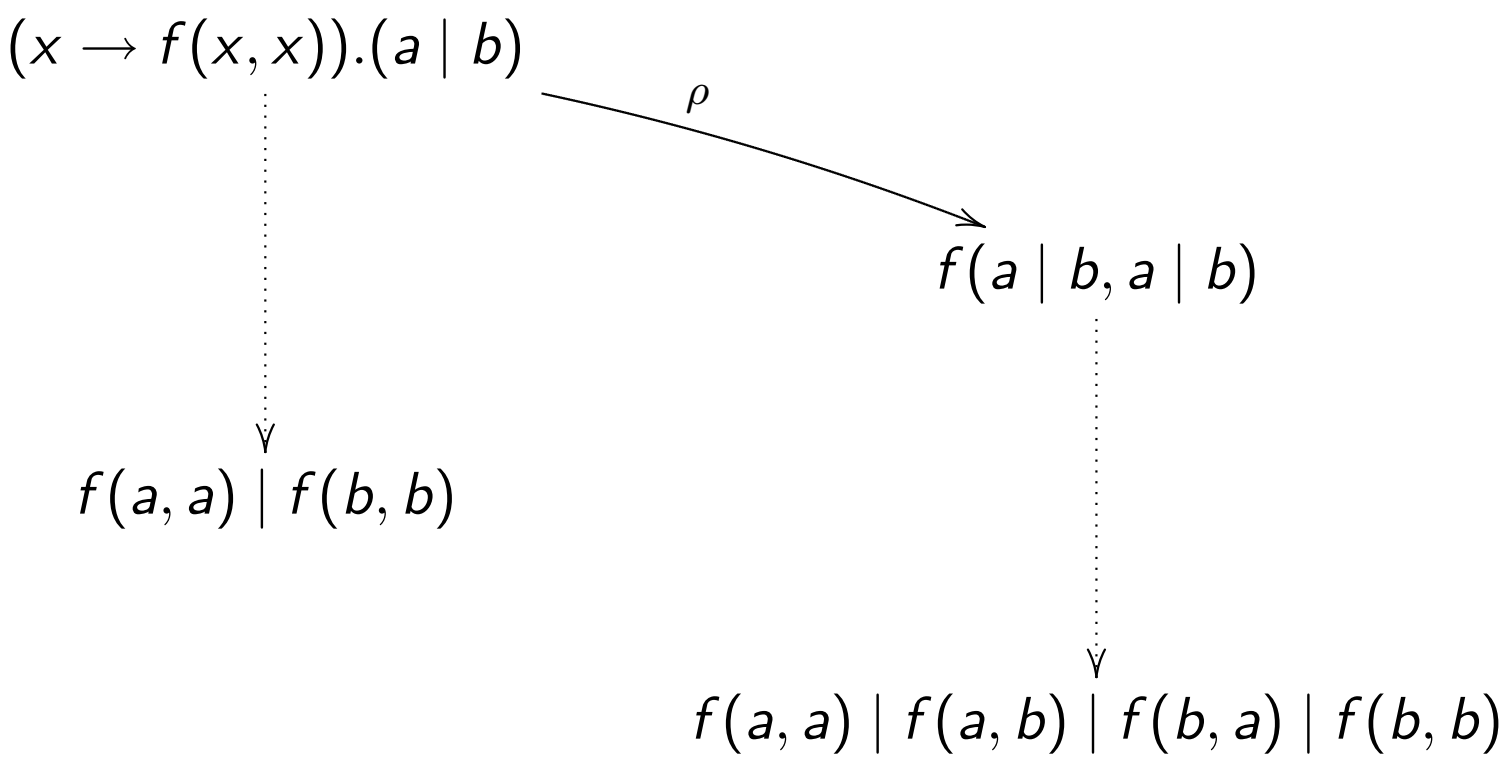

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- other solution ?

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- ...

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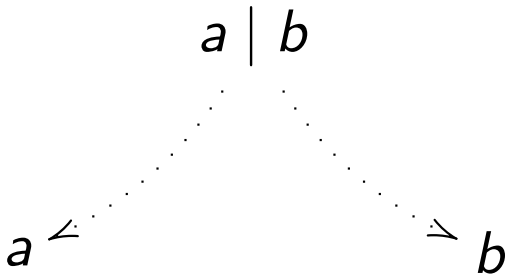

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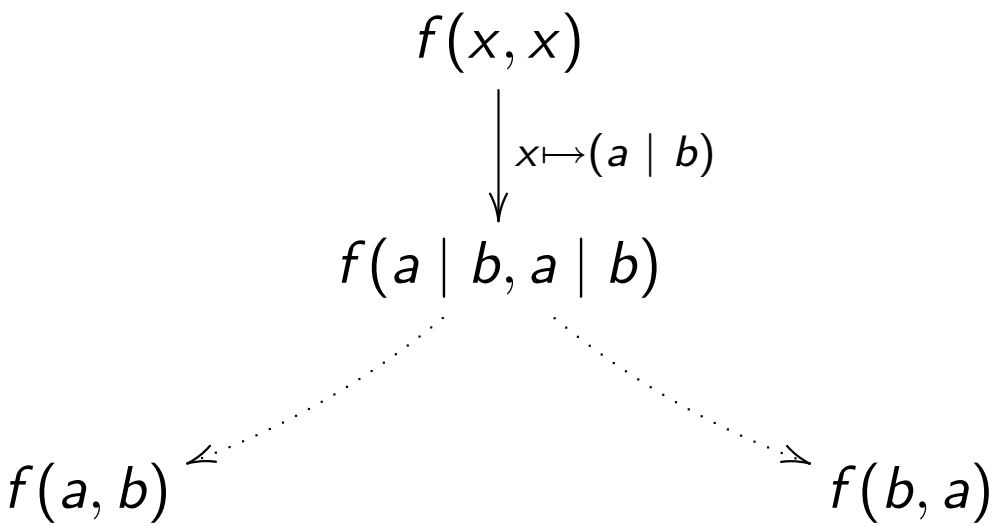
... and substituting a variable for such a term is dangerous



Values

Values $\approx$ Deterministic terms

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Value substitution

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- Then :

$$Sol(P \ll M) = \{\sigma / \sigma P =_T M \text{ and } \sigma \text{ value substitution}\}$$

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No more interference between non-linear terms and non-determinism :
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- We add the rule

$$(\gamma) \quad N.(M_1 \mid M_2) \quad \mapsto \quad (N.M_1) \mid (N.M_2)$$

Interaction between a theory and the ρ-calculus

"::" is here an associative symbol.

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$$a \mid (a :: b)$$

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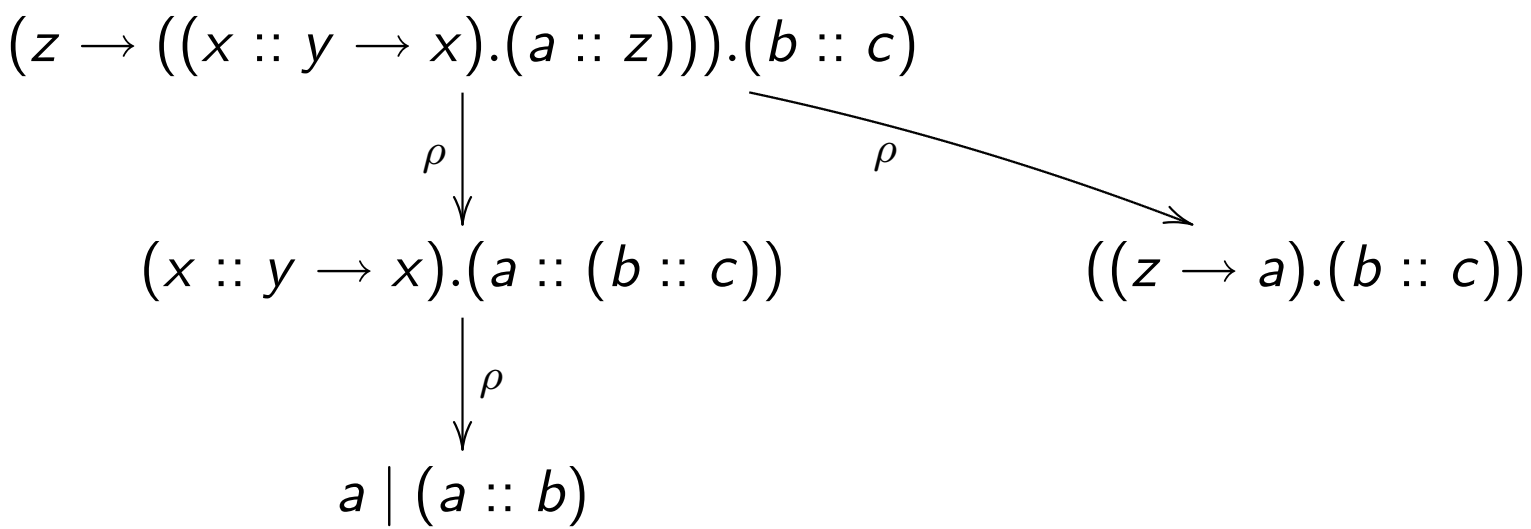

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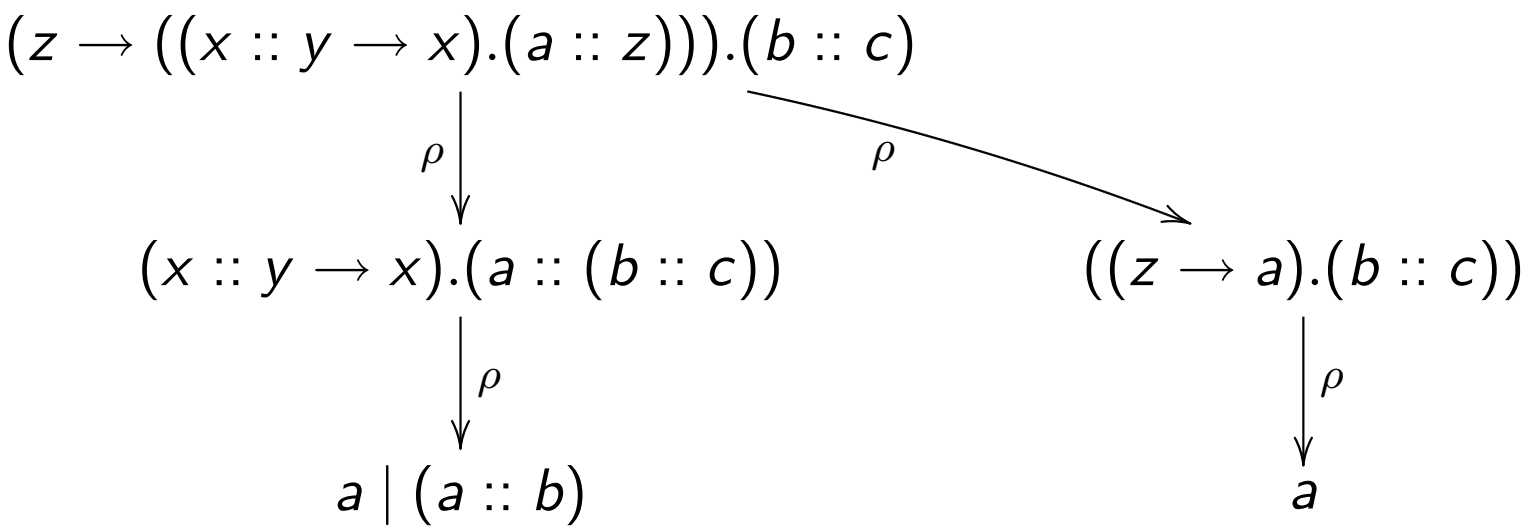

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- Solutions of a pattern matching problem can change if we substitute a variable for a value...

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  - variable are not values

rho-terms

Definition of. . .

- Patterns,
- Terms,
- Values.

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The ρ-calculus modulo

$$\rho : \quad (P \to M).V \quad \mapsto \quad \sigma_1 M \mid \ldots \mid \sigma_n M \text{ avec } \begin{cases} \{\sigma_i\}_i = \mathcal{S}ol(P \ll V) \\ n \geq 1 \end{cases}$$

$$\gamma : \quad M.(N_1 \mid N_2) \quad \mapsto \quad (M.N_1) \mid (M.N_2)$$

$$\delta : \quad (N_1 \mid N_2).V \quad \mapsto \quad (N_1.V) \mid (N_2.V)$$

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- We have no restriction on linearity or theories.
- Good expressiveness : call-by-value lambda-calculus, fixpoints, Caml

Fixpoints

let *plus* be

$$REC \longrightarrow \left(\begin{array}{l} (add.z.Y \mid \longrightarrow Y) \mid \\ (add.(s.X).Y \longrightarrow s.((REC.REC).(add.X.Y))) \end{array} \right)$$

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Disadvantage

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- it could be very long...
- ... even impossible ($T = \Omega$)

What for ?

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- Solution : constraint introduction

Introduction of constraints

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(\rho) & (P \longrightarrow M).V & \mapsto & \sigma_1 M \mid \ldots \mid \sigma_n M \\
 & & \Downarrow & \\
(\rho) & (P \longrightarrow M).N & \mapsto & M[P \ll N] \\
(eval) & M[P \ll V] & \mapsto & \sigma_1 M \mid \ldots \mid \sigma_n M
\end{array}
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(eval) & M[P \ll V] & \mapsto & \sigma_1 M \mid \ldots \mid \sigma_n M \\
 & & & \\
(delete) & M[P \ll N] & \mapsto & M \\
 & & & \text{When } FV(M) \cap dom(P) = \emptyset
\end{array}
$$

Improving

$$x[y \ll T] \quad \mapsto \quad x$$

Improving

$$\begin{aligned} x[y \ll T] &\mapsto x \\ x[f(x,y) \ll f(a,T)] &\mapsto_? a \end{aligned}$$

. . . We need explicit constraint resolution to improve completely the calculus.

Results

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- confluent
- that handles any theory
- that could be freed from strategy reduction disadvantages...

Further work. . .

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- ...on expressiveness of the ρ-calculus modulo

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- ...on explicit constraints resolution

Further work. . .

- . . . on expressiveness of the ρ-calculus modulo
- . . . on explicit constraints resolution
- . . . on extensions of the ρ-calculus modulo