Superdeduction

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Let's type the ρ-calculus

Terms:

$$p ::= \quad R(\alpha_1, \ldots, \alpha_n)$$

$$m ::= \quad R(t_1, \ldots, t_n)$$

$$t ::= \quad x \mid t\ m \mid p \to t$$

Types:

$$T ::= A \mid T \Rightarrow T$$

Let's type the ρ-calculus

For each R of arity n, we associate some type

$$P = A_1 \Rightarrow A_2 \Rightarrow \ldots \Rightarrow A_n \Rightarrow B$$

$$\frac{\Gamma, \alpha_1 : A_1, \ldots, \alpha_n : A_n \vdash t : B}{\Gamma \vdash R(\alpha_1, \ldots, \alpha_n) \to t : P}$$

$$\frac{\Gamma \vdash t : P \qquad \Gamma \vdash t_1 : A_1 \qquad \ldots \qquad \Gamma \vdash t_n : A_n}{\Gamma \vdash t\ R(t_1, \ldots, t_n) : B}$$

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Let's focus on logic

$$R \quad : \quad P = \quad (A_1 \Rightarrow A_2 \Rightarrow \ldots \Rightarrow B)$$

$$R\text{-INTRO}\ \frac{\Gamma, A_1, \ldots, A_n \vdash B}{\Gamma \vdash P}$$

$$R\text{-ELIM}\ \frac{\Gamma \vdash P \qquad \Gamma \vdash A_1 \qquad \ldots \qquad \Gamma \vdash A_n}{\Gamma \vdash B}$$

Let's focus on logic

$$R \quad : \quad P = \forall \overline{x}.(A_1 \Rightarrow A_2 \Rightarrow \ldots \Rightarrow B)$$

$$R\text{-INTRO}\ \frac{\Gamma, A_1, \ldots, A_n \vdash B}{\Gamma \vdash P}\ \overline{x} \notin \mathcal{FV}(\Gamma)$$

$$R\text{-ELIM}\ \frac{\Gamma \vdash P \qquad \Gamma \vdash A_1[\overline{t}/\overline{x}] \qquad \ldots \qquad \Gamma \vdash A_n[\overline{t}/\overline{x}]}{\Gamma \vdash B[\overline{t}/\overline{x}]}$$

A new framework for first order logic

$$\subseteq \quad : \quad A \subseteq B = \forall x.(x \in A \Rightarrow x \in B)$$

$$\subseteq\text{-INTRO}\ \frac{\Gamma, x \in A \vdash x \in B}{\Gamma \vdash A \subseteq B}$$

$$\subseteq\text{-ELIM}\ \frac{\Gamma \vdash A \subseteq B \qquad \Gamma \vdash x \in A}{\Gamma \vdash x \in B}$$

A new framework for first order logic

$$\subseteq \quad : \quad A \subseteq B = \forall x.(x \in A \Rightarrow x \in B)$$

$$\subseteq\text{-INTRO}\ \frac{\Gamma, x \in A \vdash x \in B}{\Gamma \vdash A \subseteq B}$$

"Supposing that $x \in A$, we can prove that $x \in B$ without assuming anything on x. Therefore $A \subseteq B$"

A new framework for first order logic

$$\subseteq \quad : \quad A \subseteq B = \forall x.(x \in A \Rightarrow x \in B)$$

$$\subseteq\text{-ELIM}\ \frac{\Gamma \vdash A \subseteq B \qquad \Gamma \vdash t \in A}{\Gamma \vdash t \in B}$$

"If we can prove that $A \subseteq B$ and that $t \in A$, then we can prove that $t \in B$."

Building the rules

$$R: \; P \rightarrow (A \Rightarrow B) \Rightarrow (C \wedge D)$$

$$\dfrac{\dfrac{\vdash (A \Rightarrow B) \Rightarrow (C \wedge D) \qquad \vdash A \Rightarrow B}{\vdash C \wedge D} \Rightarrow\text{-ELIM}}{\vdash C} \wedge\text{-ELIM-1}$$

Building the rules

$$R : \; P \to (A \Rightarrow B) \Rightarrow (C \wedge D)$$

$$\Rightarrow\text{-ELIM}\ \dfrac{\vdash (A \Rightarrow B) \Rightarrow (C \wedge D) \qquad \vdash A \Rightarrow B}{\wedge\text{-ELIM-1}\ \dfrac{\vdash C \wedge D}{\vdash C}}$$

$$\downarrow$$

$$R\text{-ELIM-1}\ \dfrac{\vdash P \qquad \vdash A \Rightarrow B}{\vdash C}$$

Building the rules

$$R: \ P \rightarrow (A \Rightarrow B) \Rightarrow (C \wedge D)$$

$$\dfrac{\dfrac{A \Rightarrow B \vdash C \qquad A \Rightarrow B \vdash D}{A \Rightarrow B \vdash C \wedge D}\ \wedge\text{-INTRO}}{\vdash (A \Rightarrow B) \Rightarrow (C \wedge D)}\ \Rightarrow\text{-INTRO}$$

Building the rules

$$R : \; P \to (A \Rightarrow B) \Rightarrow (C \wedge D)$$

$$\dfrac{\dfrac{A \Rightarrow B \vdash C \qquad A \Rightarrow B \vdash D}{A \Rightarrow B \vdash C \wedge D}\ \wedge\text{-INTRO}}{\vdash (A \Rightarrow B) \Rightarrow (C \wedge D)}\ \Rightarrow\text{-INTRO}$$

$$\downarrow$$

$$\dfrac{A \Rightarrow B \vdash C \qquad A \Rightarrow B \vdash D}{\vdash P}\ R\text{-INTRO}$$

Limitations of supernatural deduction

- incomplete decomposition:

$$R : P \rightarrow (A \wedge B) \Rightarrow C \quad \leadsto \quad R\text{-}\textsc{elim}\ \frac{\Gamma \vdash_{+} P \qquad \Gamma \vdash_{+} A \wedge B}{\Gamma \vdash_{+} C}$$

- restricted number of decomposed connectors:
  Only $\Rightarrow$, $\wedge$ and $\forall$

$$P \rightarrow \forall x.(A(x) \vee B(x))$$

and $$P \rightarrow \forall x.A(x) \vee \forall x.B(x)$$

lead to the same deduction rules

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  lead to the same deduction rules

- Sequent calculus presents more symmetries

Extended sequent calculus

- Rules computation : same procedure with all connectors
  - One right and one left super-rule per axiom
  - Two additional computation rules

$$\top\text{-LEFT}\ \frac{\Gamma \vdash \Delta}{\Gamma, \top \vdash \Delta} \qquad \bot\text{-RIGHT}\ \frac{\Gamma \vdash \Delta}{\Gamma \vdash \bot, \Delta}$$

- One issue : non-permutability cases of classical sequent calculus

Permutability problem

- Superdeduction = Prawitz' folding-unfolding + automated deduction
- Problems : ∀-RIGHT then ∀-LEFT or ∃-RIGHT, etc.

Permutability problem

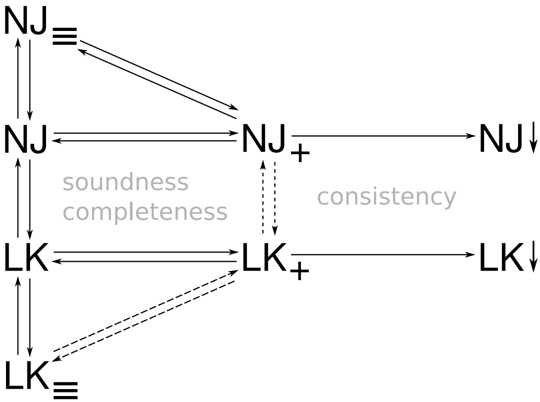
- Superdeduction = Prawitz' folding-unfolding + automated deduction
- Problems : $\forall$-RIGHT then $\forall$-LEFT or $\exists$-RIGHT, etc.
- Solution : *focussing*

$$R : P \rightarrow \forall x.(\forall y.A(x,y) \Rightarrow B(x))$$

$$\downarrow$$

$$\begin{array}{rcrcl} R & : & P & \rightarrow & \forall x.(P_1(x) \Rightarrow B(x)) \\ R_1 & : & P_1(x) & \rightarrow & \forall y.A(x,y) \end{array}$$

Translations summary



Strong normalization result

For supernatural deduction systems:

- Curry-Howard correspondance with a simple ρ-calculus
- Subject reduction + Strong normalization
- Consistency of supernatural deduction

Deduction modulo

Modulo

$$\forall\text{-RIGHT}\ \frac{\Gamma \vdash P(x_0), \Delta}{\Gamma \vdash Q(x), \Delta}\ Q \equiv_{\mathcal{R}} \forall x.P(x)$$

$$\forall\text{-RIGHT}\ \frac{\vdash x_0 \in A \Rightarrow x_0 \in B}{\vdash A \subseteq B}$$

Superdeduction

$$\subseteq\text{-RIGHT}\ \frac{\Gamma, x_0 \in X \vdash x_0 \in Y, \Gamma}{\Gamma \vdash X \subseteq Y, \Gamma}$$

$$\subseteq\text{-RIGHT}\ \frac{x_0 \in A \vdash x_0 \in B}{\vdash A \subseteq B}$$

Deduction modulo

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$$\forall\text{-RIGHT}\ \frac{\Gamma \vdash P(x_0), \Delta}{\Gamma \vdash Q(x), \Delta}\ Q \equiv_{\mathcal{R}} \forall x.P(x)$$

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Superdeduction

$$\subseteq\text{-RIGHT}\ \frac{\Gamma, x_0 \in X \vdash x_0 \in Y, \Gamma}{\Gamma \vdash X \subseteq Y, \Gamma}$$

$$\subseteq\text{-RIGHT}\ \frac{x_0 \in A \vdash x_0 \in B}{\vdash A \subseteq B}$$

Application : arithmetic

- Natural numbers definition $\rightarrow$ induction principle
- "cleaning" the rule:

$$\in_{\mathbb{N}}:\ n \in \mathbb{N} \ \rightarrow\ \forall P.(0 \in P \Rightarrow \forall m.(m \in P \Rightarrow S(m) \in P) \Rightarrow n \in P)$$

$$\downarrow$$

$$\begin{array}{rcccl} \in_{\mathbb{N}} & : & n \in \mathbb{N} & \rightarrow & \forall P.(0 \in P \Rightarrow H(P) \Rightarrow n \in P) \\ \mathit{hered} & : & H(P) & \rightarrow & \forall m.(m \in P \Rightarrow S(m) \in P) \end{array}$$

Application : arithmetic

New deduction rules for *hered*:

$$\textit{hered}\text{-}\textsc{left}\ \frac{\Gamma \vdash_+ m \in P, \Delta \qquad \Gamma, S(m) \in P \vdash_+ \Delta}{\Gamma, H(P) \vdash_+ \Delta}$$

$$\textit{hered}\text{-}\textsc{right}\ \frac{\Gamma, m \in P \vdash_+ S(m) \in P, \Delta}{\Gamma \vdash_+ H(P), \Delta}$$

Application : arithmetic

New deduction rules fo $\in_{\mathbb{N}}$:

$$\in_{\mathbb{N}}\text{-LEFT}\ \frac{\Gamma \vdash_+ 0 \in P, \Delta \qquad \Gamma \vdash_+ H(P), \Delta \qquad \Gamma, n \in P \vdash_+ \Delta}{\Gamma, n \in \mathbb{N} \vdash_+ \Delta}$$

$$\in_{\mathbb{N}}\text{-RIGHT}\ \frac{0 \in P, H(P) \vdash_+ n \in P, \Delta}{\Gamma \vdash_+ n \in \mathbb{N}, \Delta}$$

Application : arithmetic

New deduction rules fo $\in_{\mathbb{N}}$:

$$\in_{\mathbb{N}}\text{-LEFT}\ \frac{\Gamma \vdash_+ 0 \in P, \Delta \qquad \Gamma \vdash_+ H(P), \Delta \qquad \Gamma, n \in P \vdash_+ \Delta}{\Gamma, n \in \mathbb{N} \vdash_+ \Delta}$$

$$\in_{\mathbb{N}}\text{-RIGHT}\ \frac{0 \in P, H(P) \vdash_+ n \in P, \Delta}{\Gamma \vdash_+ n \in \mathbb{N}, \Delta}$$

Induction principle

Prototype

Lemuridae : a proof assistant for superdeduction

- Rewrite rules on terms and propositions
- Proof building in th extendible sequent calculus
- Interactive matching rules presentation
- Automatic tactics