Higher-Order Termination
From Kruskal to Computability

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Workshop RHO 2006, London, october 17, 2006

Outline

Higher-order algebras [Jouannaud, Rubio, JACM to appear]

Types, signatures and terms

- $\mathcal{S}$: set of *sort symbols* of a fixed arity, denoted by $s : *^n \Rightarrow *$

- $$\mathcal{T}_\mathcal{S} := s(\mathcal{T}_\mathcal{S}^n) \mid (\mathcal{T}_\mathcal{S} \rightarrow \mathcal{T}_\mathcal{S})$$
  $$\text{for } s : *^n \Rightarrow * \;\in \mathcal{S}$$

- $$\mathcal{T} := \mathcal{X} \mid (\lambda \mathcal{X}.\mathcal{T}) \mid @(\mathcal{T}, \mathcal{T}) \mid \mathcal{F}(\mathcal{T}, \ldots, \mathcal{T}).$$

  We will sometimes write $\mathcal{T}(\mathcal{T})$ for $@(\mathcal{T}, \mathcal{T})$.

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Typing rules

Functions:

$$\frac{f : \sigma_1 \times \ldots \times \sigma_n \Rightarrow \sigma \qquad \Gamma \vdash t_1 : \tau_1 \ \ldots \ \Gamma \vdash t_n : \tau_n \qquad \theta = mgu(\sigma_1 = \tau_1 \ \& \ \ldots \ \& \ \sigma_n = \tau_n)}{\Gamma \vdash f(t_1, \ldots, t_n) : \sigma}$$

Variables:

$$\frac{x : \sigma \in \Gamma}{\Gamma \vdash x : \sigma}$$

Abstraction:

$$\frac{\Gamma \cup \{x : \sigma\} \vdash t : \tau}{\Gamma \vdash (\lambda x : \sigma . t) : \sigma \rightarrow \tau}$$

Application:

$$\frac{\Gamma \cup \{x : \sigma\} \vdash s : \sigma \rightarrow \tau \qquad \Gamma \vdash t : \sigma}{\Gamma \vdash @(s, t) : \tau}$$

Higher-order rules, first-order pattern matching

$$
\begin{array}{rcl}
\mathbb{N}, \alpha & : & * \\
0, x & : & \mathbb{N} \\
s & : & \mathbb{N} \Rightarrow \mathbb{N} \\
rec & : & \mathbb{N} \times \alpha \times (\mathbb{N} \rightarrow \alpha \rightarrow \alpha) \Rightarrow \alpha \\
U & : & \alpha \\
X & : & \mathbb{N} \rightarrow \alpha \rightarrow \alpha
\end{array}
$$

$$
\begin{array}{rcl}
rec(0, U, X) & \rightarrow & U \\
rec(s(x), U, X) & \rightarrow & @(X, x, rec(x, U, X))
\end{array}
$$

Higher-order rules, first-order pattern matching

$$
\begin{array}{lcl}
\alpha & : & * \\
0, x & : & Ord \\
s & : & Ord \to Ord \\
lim & : & (\mathbb{N} \to Ord) \Rightarrow Ord \\
rec & : & Ord \times \alpha \times (Ord \to \alpha \to \alpha) \times ((\mathbb{N} \to Ord) \to (\mathbb{N} \to \alpha) \Rightarrow \alpha) \\
 & & \to \alpha \\
F & : & \mathbb{N} \to Ord \\
U & : & \alpha \\
X & : & Ord \to \alpha \to \alpha \\
W & : & (\mathbb{N} \to Ord) \to (\mathbb{N} \to \alpha) \to \alpha
\end{array}
$$

$$
\begin{array}{lcl}
rec(0, U, X, W) & \to & U \\
rec(s(x), U, X, W) & \to & @(X, x, rec(x, U, X, W)) \\
rec(lim(F), U, X, W) & \to & @(W, F, \lambda n.rec(@(F, n), U, X, W))
\end{array}
$$

Goal

Automate
strong normalization
proofs

Tait's method

Language

- Simple type discipline
- One rewrite schema:

$$@(\lambda x.u, v) \rightarrow u\{x \mapsto v\}$$

$[\![\sigma]\!]$, the *computability predicate* of type σ s.t.:
(i) computable terms are strongly normalizing;
(ii) reducts of computable terms are computable;
(iii) a neutral term u is computable iff all its reducts are computable;
(iv) $u : \sigma \rightarrow \tau$ is computable iff so is $@(u, v)$ for all computable v;
(v) (optionnal) $\lambda x.u$ is computable iff so is $u\{x \mapsto v\}$ for all computable v.

Except (v), no explicit mention of β-reduction.

Examples of computability predicates

- Basic types: there are two possibilities

  $s : \sigma \in [\![\sigma]\!]$ iff s is strongly normalizing
  or
  $s : \sigma \in [\![\sigma]\!]$ iff $\forall t : \tau$ s.t. $s \longrightarrow t$ then $t \in [\![\tau]\!]$

- Functional types:
  $s : \theta \rightarrow \tau \in [\![\sigma \rightarrow \tau]\!]$ iff $@(s, u) : \tau \in [\![\tau]\!]$ for every $u : \theta \in [\![\theta]\!]$.

Main Lemma

Given term s and computable substitution γ, then $s\gamma$ is computable.
By induction on the structure of terms.

1. $s \in \mathcal{X}$. $s\gamma$ computable by assumption.
2. $s = @(u, v)$. $u\gamma$ and $v\gamma$ are computable by induction hypothesis, hence $s\gamma = @(u\gamma, v\gamma)$ is computable by computability property (iv).
3. $s = \lambda x.u$. By property (v), $s\gamma = \lambda x.u\gamma$ is computable iff $u\gamma\{x \mapsto v\} = u(\gamma \cup \{x \mapsto v\})$ is computable for all computable v. We conclude by induction hypothesis.

Recursive path ordering

Recursive path ordering: $s \succ_{rpo} t$ iff

1. $s = f(\overline{s})$ with $f \in \mathcal{F}$, and $u \underset{rpo}{\succeq} t$ for some $u \in \overline{s}$
2. $s = f(\overline{s})$ with $f \in \mathcal{F}$, and $t = g(\overline{t})$ with $f >_{\mathcal{F}} g$, and A
3. $s = f(\overline{s})$ and $t = g(\overline{t})$ with $f =_{\mathcal{F}} g$, and A and $\overline{s} \; (\underset{rpo}{\succ})_{stat_f} \; \overline{t}$

$$\text{where } \begin{cases} s \succeq_{rpo} t \text{ iff } s \succ_{rpo} t \text{ or } s = t \\ A = \forall v \in \overline{t}. f(\overline{s}) \succ_{horpo} v \end{cases}$$

Computability is defined as strong normalization, implying all computability properties trivially. We add a new computability property:

(vi) Let $f \in \mathcal{F}_n$ and $\overline{s}$ be computable terms. Then $f(\overline{s})$ is computable.

Tait's strong normalization proof of RPO

First (vi): $\overline{s}$ computable implies $f(\overline{s})$ computable.

The restriction of $\succ_{rpo}$ to terms smaller than or equal to the terms in $\overline{s}$ w.r.t. $\succ_{rpo}$ is a well-founded ordering which we use for building an outer induction on the pairs $(f, \overline{s})$ ordered by $(>_{\mathcal{F}}, (\succ_{rpo})_{stat_f})_{lex}$.

We now show that $f(\overline{s})$ is computable by proving that t is computable for all t such that $f(\overline{s}) \succ_{rpo} t$. This property is itself proved by an inner induction on $\mid t \mid$, and by case analysis upon the proof that $f(\overline{s}) \succ_{rpo} t$.

Property (vi)

1. subterm: $\exists u \in \overline{s}$ such that $u \succ_{rpo} t$. By assumption, u is computable. Reduct t too.
2. precedence: $t = g(\overline{t})$, $f >_{\mathcal{F}} g$, and $s \succ_{rpo} \overline{t}$. By inner induction, $\overline{t}$ is computable. By outer induction, $g(\overline{t}) = t$ is computable.
3. status: $t = g(\overline{t})$ with $f =_{\mathcal{F}} g \in Lex$, $\overline{s}(\succ_{rpo})_{lex}\overline{t}$, and $s \succ_{rpo} \overline{t}$. By inner induction, $\overline{t}$ is computable. By outer induction, $g(\overline{t}) = t$ is computable. $\square$

Main Lemma

We prove by induction on the structure of terms that every term $t = f(\bar{t})$ is computable.
By induction hypothesis, $\bar{t}$ is computable. By property (vi), t is computable.

The well-foundedness of $\succ_{rpo}$ follows by Property (i).

General Schema

Closure and General Schema

The *computability closure* $\mathcal{CC}(t = f(\bar{t}))$, with $f \in \mathcal{F}$, is the set $\mathcal{CC}(t, \emptyset)$, s.t. $\mathcal{CC}(t, \mathcal{V})$, with $\mathcal{V} \cap \mathcal{V}ar(t) = \emptyset$, is the smallest set of typable terms containing all variables in $\mathcal{V}$ and terms in $\bar{t}$, closed under:

1. basic type subterm; application; abstraction;
2. precedence: let $f >_{\mathcal{F}} g$, and $\bar{s} \in \mathcal{CC}(t, \mathcal{V})$; then $g(\bar{s}) \in \mathcal{CC}(t, \mathcal{V})$;
3. recursive call: let $f(\bar{s})$ be a term s.t. terms in $\bar{s}$ belong to $\mathcal{CC}(t, \mathcal{V})$ and $\bar{t}(\longrightarrow_{\beta \cup \rhd})_{stat_f} \bar{s}$; then $g(\bar{s}) \in \mathcal{CC}(t, \mathcal{V})$ for every $g =_{\mathcal{F}} f$;
4. reduction: let $u \in \mathcal{CC}(t, \mathcal{V})$, and $u \longrightarrow_{\beta \cup \rhd} v$; then $v \in \mathcal{CC}(t, \mathcal{V})$.

General schema [Blanqui, Jouannaud and Okada, TCS 2001]

We say that a rewrite system R satisfies the *general schema* if

$$R = \{f(\bar{l}) \rightarrow r \mid r \in \mathcal{CC}(f(\bar{l}))\}$$

We now consider computability with respect to the rewrite relation $\longrightarrow_R \cup \longrightarrow_\beta$, and add the computability property (vii) whose proof can be easily adapted from the previous one. We can then add a new case in Tait's Main Lemma, for terms headed by an algebraic function symbol.

Conclusion: $\longrightarrow_\beta \cup \longrightarrow_R$ is SN.

Example : System T

$$rec(s(x), U, X) \rightarrow @(X, x, rec(x, U, X))$$

Higher-Order Recursive Path Ordering: Ingredients

- A quasi-ordering on types $\geq_{\mathcal{T}_S}$ called *the type ordering* s.t.
  (i) $>_{\mathcal{T}_S}$ is well-founded;
  (ii) *Arrow preservation*: $\tau \to \sigma =_{\mathcal{T}_S} \alpha$ iff $\alpha = \tau' \to \sigma',\ \tau' =_{\mathcal{T}_S} \tau$ and $\sigma =_{\mathcal{T}_S} \sigma'$;
  (iii) *Arrow decreasingness*: $\tau \to \sigma >_{\mathcal{T}_S} \alpha$ implies $\sigma \geq_{\mathcal{T}_S} \alpha$ or $\alpha = \tau' \to \sigma', \tau' =_{\mathcal{T}_S} \tau$ and $\sigma >_{\mathcal{T}_S} \sigma'$;
  (iv) *Arrow monotonicity*: $\tau \geq_{\mathcal{T}_S} \sigma$ implies $\alpha \to \tau \geq_{\mathcal{T}_S} \alpha \to \sigma$ and $\tau \to \alpha \geq_{\mathcal{T}_S} \sigma \to \alpha$;

  Example: RPO with restricted subterm for $\to$

HORPO's Ingredients

- A quasi-ordering $\geq_{\mathcal{F}}$ on $\mathcal{F}$, called the *precedence*, such that $>_{\mathcal{F}}$ is well-founded.
- A *status* $stat_f \in \{Mul, Lex\}$ for every symbol $f \in \mathcal{F}$.

HORPO's Definition: $s \succ_{horpo} t$ iff $\sigma \geq_{\mathcal{T}_S} \tau$ and

1. $s = f(\overline{s})$ with $f \in \mathcal{F}$, and $u \underset{horpo}{\succeq} t$ for $u \in \overline{s}$
2. $s = f(\overline{s})$ with $f \in \mathcal{F}$, and $t = g(\overline{t})$ with $f >_{\mathcal{F}} g$, and A
3. $s = f(\overline{s})$ and $t = g(\overline{t})$ with $f =_{\mathcal{F}} g$, and A and $\overline{s} \; (\underset{horpo}{\succ})_{stat_f} \; \overline{t}$

$$\text{where } \begin{cases} s \succeq_{horpo} t \text{ iff } s \succ_{horpo} t \text{ or } s =_{\alpha} t \\ A = \forall v \in \overline{t}.s \succ_{horpo} v \text{ or } \exists u \in \overline{s}.u \succeq_{horpo} v \end{cases}$$

4. $s = f(\overline{s})$ with $f \in \mathcal{F}$, $t = @(\overline{t})$ and A
5. $s = f(\overline{s})$ with $f \in \mathcal{F}$, $t = \lambda x : \alpha.v$ with $x \notin \mathcal{V}ar(v)$ and $s \underset{horpo}{\succ} v$

Higher-Order Recursive Path Ordering : Definition

6. $s = @(s_1, s_2)$, and $s_1 \underset{horpo}{\succeq} t$ or $s_2 \underset{horpo}{\succeq} t$
7. $s = @(\overline{s})$, $t = @(\overline{t})$, and $\overline{s}(\underset{horpo}{\succ})_{mul}\ \overline{t}$
8. $s = @(\lambda x : \alpha.u, v)$ and $u\{x \mapsto v\} \underset{horpo}{\succeq} t$
9. $s = \lambda x : \alpha.u$ with $x \notin \mathcal{V}ar(t)$, and $u \underset{horpo}{\succeq} t$
10. $s = \lambda x : \alpha.u,\ t = \lambda x : \beta.v$, $\alpha =_{\mathcal{T}_S} \beta$, and $u \underset{horpo}{\succ} v$
11. $s = \lambda x : \alpha.@(u, x),\ x \notin \mathcal{V}ar(u)$ and $u \underset{horpo}{\succeq} t$

Example: simple proof of system T

$$rec(s(x), U, X) \to @(X, x, rec(x, U, X))$$

HORPO and Closure

Combining HORPO and closure

We change the subterm case:

1. $s = f(\overline{s})$ with $f \in \mathcal{F}$ and $u \underset{horpo}{\succeq} t$ for $u \in \overline{s}$

in

$s = f(\overline{s})$ with $f \in \mathcal{F}$ and $u \underset{horpo}{\succeq} t$ for $u \in \mathcal{CC}(f(\overline{s}))$

Drawbacks:

1. Decidability of HORPO is lost;
2. There are many repetitions;
3. Type checking is no much help, but a lot of burden;
4. Treatment of abstractions remains weak.

New HORPO with integrated closure mechanism

Ingredients:

1. A set of strictly positive inductive types inducing an accessibility relationship $\overline{s} \trianglerighteq_{acc} v$ such that $v \in \overline{u}$ or v is accessible from $u \in \overline{s}$
2. a precedence on function symbols
3. a congruence on types
4. $s \succ^X t$ for the main ordering
5. $s : \sigma \succ^X_{\mathcal{T}_S} t : \tau$ for $s \succ^X t$ and $\sigma =_{\mathcal{T}_S} \tau$
6. $l \succ^{\emptyset}_{\mathcal{T}_S} r$ as initial call for each $l \to r \in R$

Definition : $s \succ^X t$ iff

Case 1: $s = f(\overline{s})$ with $f \in \mathcal{F}$ and $t \in X$ or

1. $u \succeq^X_{\mathcal{T}_S} t$ for some u such that $\overline{s} \trianglerighteq_{acc} u$
2. $t = g(\overline{t})$ with $f >_{\mathcal{F}} g \in \mathcal{F} \cup \{@\}$ and $s \succ^X \overline{t}$
3. $t = g(\overline{t})$ with $f =_{\mathcal{F}} g \in \mathcal{F}$ and $s \succ^X \overline{t}$ and $\overline{s}(\succ^X_{\mathcal{T}_S})_{stat_f}\overline{t}$
4. $t = \lambda x.u$ with $x \notin X$ and $f(\overline{s}) \succ^{X \cup \{x\}} u$

Case 2: $s = @(v, w)$ and

1. $t = @(u, r)$ and $(v, w)(\succ^X_{\mathcal{T}_S})_{mon}(u, r)$
2. $v = \lambda x.u$ and $u\{x \mapsto w\} \succ^X t$

Case 3: $s = \lambda x : \alpha.u$ and

1. $t = \lambda x : \beta.v$, $x \notin X$, $\alpha =_{\mathcal{T}_S} \beta$ and $u \succ^{X \cup \{x\}} v$
2. $u = @(v, x)$, $x \notin \mathcal{V}ar(v)$ and $v \succ^X t$

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2. $u = @(v, x)$, $x \notin \mathcal{V}ar(v)$ and $v \succ^X t$

Brouwer's ordinals

$lim : (\mathbb{N} \rightarrow Ord) \Rightarrow Ord \qquad F : \mathbb{N} \rightarrow Ord \qquad n : \mathbb{N}$

$rec : Or \times \alpha \times (Or \rightarrow \alpha \rightarrow \alpha) \times ((\mathbb{N} \rightarrow Or) \rightarrow (\mathbb{N} \rightarrow \alpha) \rightarrow \alpha) \Rightarrow \alpha$

1. $rec(lim(F), U, X, W) \succ^{\emptyset}_{\mathcal{T}_S} @(W, F, \lambda n.rec(@(F, n), U, X, W))$ yields 2 subgoals:
2. $\alpha =_{\mathcal{T}_S} \alpha$ which is trivially satisfied, and
3. $rec(lim(F), U, X, W) \succ^{\emptyset} \{W, F, \lambda n.rec(@(F, n), U, X, W)\}$ which simplifies to:
4. $rec(lim(F), U, X, W) \succ^{\emptyset} W$ which succeeds by Case 1.1,
5. $rec(lim(F), U, X, W) \succ^{\emptyset} F$, which succeeds by Case 1.1,
6. $rec(lim(F), U, X, W) \succ^{\emptyset} \lambda n.rec(@(F, n), U, X, W)$ yields
7. $rec(lim(F), U, X, W) \succ^{\{n\}} rec(@(F, n), U, X, W)$ yields
8. $\{lim(F), U, X, W\}(\succ^{\{n\}}_{\mathcal{T}_S})_{mul}\{@(F, n), U, X, W\}$, hence
9. $lim(F) \succ^{\{n\}}_{\mathcal{T}_S} @(F, n)$ whose type-check succeeds, and yields
10. $lim(F) \succ^{\{n\}} F$ which succeeds by Case 1.2, and
11. $lim(F) \succ^{\{n\}} n$ which succeeds by Case 1.
12. $rec(lim(F), U, X, W) \succ^{\{n\}} \{@(F, n), U, X, W\}$, our remaining goal, succeeds easily by Cases 1.2, 1 and 2.1

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8. $\{lim(F), U, X, W\}(\succ^{\{n\}}_{\mathcal{T}_S})_{mul}\{@(F, n), U, X, W\}$, hence
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Conclusion

Achievements: A quite powerful powerful which adapts easily to higher-order rewriting based on higher-order pattern matching. See [Jouannaud and Rubio, RTA'2006]

Remaining problems:

- Use term interpretations instead of a precedence on function symbols;
- Integrate AC;
- Generalization to the Calculus of Inductive Constructions;
- Develop the tool (see our Web page).

Acknowledgments: to Mitsuhiro Okada for our long standing collaboration on these matters.