Natural Deduction with Generalized Introductions and Eliminations

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Deduction modulo

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From a rewrite rule to a deduction rule

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Some results (well, conjectures)

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From a rewrite rule to a deduction rule

Some results (well, conjectures)

A proof-term language with patterns

Natural deduction

$$(Ax)\ \frac{}{\Gamma, \phi \vdash \phi} \qquad (\Rightarrow I)\ \frac{\Gamma, \psi \vdash \phi}{\Gamma \vdash \psi \Rightarrow \phi} \qquad (\Rightarrow E)\ \frac{\Gamma \vdash \psi \Rightarrow \phi \qquad \Gamma \vdash \psi}{\Gamma \vdash \phi}$$

$$(\forall I)\ \frac{\Gamma \vdash \phi}{\Gamma \vdash \forall x.\phi}\ (x \notin \mathcal{FV}ar(\Gamma)) \qquad (\forall E)\ \frac{\Gamma \vdash \forall x.\phi}{\Gamma \vdash \phi[x := t]}$$

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- The whole theory has to be given in Γ.
- Proofs are represented as λ-terms :

axiom	introduction	elimination
variable	abstraction	application

A running example

The theory $\mathcal{T}$ contains at least $\left\{ \begin{array}{l} X \subseteq Y \Leftrightarrow \forall x(x \in X \Rightarrow x \in Y) \\ \forall x(x \in \emptyset \Rightarrow \bot) \end{array} \right.$

$$
(\Rightarrow E)\ \dfrac{(\wedge E)\ \dfrac{\mathcal{T} \vdash \ldots}{\mathcal{T} \vdash \forall x(x \in \emptyset \Rightarrow x \in A) \Rightarrow \emptyset \subseteq A} \qquad \mathcal{T} \vdash \forall x(x \in \emptyset \Rightarrow x \in A)}{\mathcal{T} \vdash \emptyset \subseteq A}
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Deduction modulo

Let $\mathcal{R}$ be a rewriting system that rewrites:

- terms to terms (*e.g.* $0 + x \;\rightarrow\; x$);
- atomic propositions to propositions (*e.g.* $x * y = 0 \;\rightarrow\; x = 0 \vee y = 0$).

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Every deduction rule is considered modulo $\cong$:

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A large part of the theory can (or should) be represented in $\mathcal{R}$.

A running example

The context is empty and $\mathcal{R} = \left\{ \begin{array}{rcl} X \subseteq Y & \longrightarrow & \forall x(x \in X \Rightarrow x \in Y) \\ x \in \emptyset & \longrightarrow & \bot \end{array} \right.$

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$$\vdash_{\cong} \emptyset \subseteq A$$

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$$x \in \emptyset \cong \bot \qquad \emptyset \subseteq A \cong \ldots$$

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\qquad
\begin{array}{l} x \in \emptyset \cong \bot \\ \\ \\ \emptyset \subseteq A \cong \dots \end{array}
$$

In standard ND the proof is $\pi_1\, H_1\, (\lambda x.\lambda\alpha.botelim(H_2\, x\, \alpha))$
Here the proof is shorter but less significant: $\lambda x.\lambda\alpha.botelim(\alpha)$

A more intuitive proof mechanism

Why not directly introduce predicate symbols appearing the goal ?
Let us consider some new rules:

$$(\subseteq I)\ \frac{\Gamma, x \in X \vdash x \in Y}{\Gamma \vdash X \subseteq Y}\ x \notin \mathcal{FV}ar(\Gamma) \qquad (\emptyset E)\ \frac{\Gamma \vdash x \in \emptyset}{\Gamma \vdash \phi}$$

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The proof is one step shorter than in NDM and bears some resemblance with an "old-school" mathematic style.

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With $P \;\longrightarrow\; (Q \wedge R) \vee S$ the new rules are:

$$(P\ I_l)\ \frac{\Gamma \vdash Q \qquad \Gamma \vdash R}{\Gamma \vdash P} \qquad (P\ I_r)\ \frac{\Gamma \vdash S}{\Gamma \vdash P} \qquad (P\ E)\ \frac{\Gamma \vdash P \qquad \Gamma, Q \wedge R \vdash U \qquad \Gamma, S \vdash U}{\Gamma \vdash U}$$

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$$(P\,I_l)\;\frac{\Gamma \vdash Q \qquad \Gamma \vdash R}{\Gamma \vdash P} \qquad (P\,I_r)\;\frac{\Gamma \vdash S}{\Gamma \vdash P} \qquad (P\,E)\;\frac{\Gamma \vdash P \qquad \Gamma, Q \wedge R \vdash U \qquad \Gamma, S \vdash U}{\Gamma \vdash U}$$

The discrepancy between $(P\,I_l)$ and the second assumption of $(P\,E)$ may ruin cut elimination, and suggests further decomposition:

$$(P\,E)\;\frac{\Gamma \vdash P \qquad \Gamma, Q, R \vdash U \qquad \Gamma, S \vdash U}{\Gamma \vdash U}$$

Conservativity

Every defined predicate is provably equivalent to its definition:

$$
(P\ I)\ \dfrac{(K\ E)\ \dfrac{\dfrac{(Ax)\ \overline{}}{\vdots}}{def \vdash H_1} \quad \ldots \quad \dfrac{\vdots}{def \vdash H_n}}{def \vdash P}
\qquad
(K\ I)\ \dfrac{\dfrac{(P\ E)\ \dfrac{P, \Gamma \vdash P \quad \ldots \quad P, \Gamma \vdash \gamma}{}}{\vdots}}{P \vdash def}
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Thus, a generalized deduction system is correct and complete if and only if the corresponding rewrite system $\mathcal{R}$ gives a correct and complete deduction modulo.

Cut elimination

A new notion of cut appears for each defined predicate:

$$
\dfrac{\dfrac{\vdots^{\mathcal{D}_2}}{\Gamma \vdash t \in X} \qquad (\subseteq I)\,\dfrac{\dfrac{\vdots^{\mathcal{D}_1}}{\Gamma, x \in X \vdash x \in Y}}{\Gamma \vdash X \subseteq Y}\,(x \notin \mathcal{FV}ar(\Gamma))}{\Gamma \vdash t \in Y}\,(\subseteq E)
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$$\dfrac{\dfrac{\vdots^{\mathcal{D}_2} \qquad \ldots \qquad \vdots^{\mathcal{D}_2}}{\vdots^{\mathcal{D}_1}}}{\Gamma \vdash t \in Y}$$

It boils down to cut elimination of the connectives appearing in the definition of the predicate.

Cut elimination (continued)

If $P \;\rightarrow\; \phi_1 \, K \, \phi_2$, should $(P\,I)$ $(K\,E)$ and $(K\,I)$ $(P\,E)$ be considered as cuts ?

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- For disjunctive connectives they form part of the definition equivalence proof.

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With $P \;\longrightarrow Q \vee R$

$$(P\,E)\;\dfrac{P \vdash P \qquad (\vee I)\;\dfrac{P, Q \vdash Q}{P, Q \vdash Q \vee R} \qquad (\vee I)\;\dfrac{P, R \vdash R}{P, R \vdash Q \vee R}}{P \vdash Q \vee R}$$

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If $P \ \rightarrow \ \phi_1 \, K \, \phi_2$, should $(P\,I)$ $(K\,E)$ and $(K\,I)$ $(P\,E)$ be considered as cuts ?

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With $P \ \rightarrow Q \vee R$

$$(P\,E)\ \dfrac{P \vdash P \qquad (\vee I)\ \dfrac{P, Q \vdash Q}{P, Q \vdash Q \vee R} \qquad (\vee I)\ \dfrac{P, R \vdash R}{P, R \vdash Q \vee R}}{P \vdash Q \vee R}$$

$$(\vee E)\ \dfrac{Q \vee R \vdash Q \vee R \qquad (P\,I)\ \dfrac{Q \vee R, Q \vdash Q}{Q \vee R, Q \vdash P} \qquad (P\,I)\ \dfrac{Q \vee R, R \vdash R}{Q \vee R, R \vdash P}}{Q \vee R \vdash P}$$

A proposal for proof terms

How to define proof-terms ?

- Add *ad-hoc* constructions in the language for each defined predicate

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- Use the λ-abstraction and store multiple assumptions and witnesses in patterns

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$$(\subseteq I)\ \frac{\Gamma, \alpha : x \in X \vdash \pi : x \in Y}{\Gamma \vdash \lambda_{\subseteq}(x, \alpha).\pi : X \subseteq Y} \qquad (\subseteq E)\ \frac{\Gamma \vdash \pi : X \subseteq Y \qquad \Gamma \vdash \pi' : t \in X}{\Gamma \vdash \pi\ {}_{\subseteq}(t, \pi') : t \in Y}$$

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The reduction $(\lambda P(\bar{x}, \bar{\alpha}).\pi)\, P(\bar{t}, \bar{\pi'}) \ \mapsto\ \pi'[\overline{x := t}, \overline{\alpha := \pi}]$ models cut elimination.

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Is there an interest for strong normalization ?

Unsound rules

It is well-known that the rewrite rule $R \;\longrightarrow\; R \Rightarrow \bot$ gives an unsound deduction modulo.
Its associated introduction and elimination rules are

$$(R\,I)\;\frac{\Gamma, R \vdash \bot}{\Gamma \vdash R} \qquad\qquad (R\,E)\;\frac{\Gamma \vdash R \qquad \Gamma \vdash R}{\Gamma \vdash \bot}$$

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and the (shortest) proof of $\vdash \bot$ has the proof-term

$$\big(\lambda R(\alpha).\alpha\, R(\alpha)\big)\ R\big(\lambda R(\alpha).\alpha\, R(\alpha)\big)$$

Curiosities

- Proof-terms with patterns for the usual connectives

$$(\wedge I)\ \frac{\Gamma \vdash \pi : \phi \qquad \Gamma \vdash \pi' : \psi}{\Gamma \vdash \wedge(\pi, \pi') : \phi \wedge \psi} \qquad (\wedge E_l)\ \frac{\Gamma \vdash \pi : \phi \wedge \psi}{\Gamma \vdash (\lambda \wedge (x, y).x)\pi : \phi}$$

$$(\vee I_l)\ \frac{\Gamma \vdash \pi : \phi}{\Gamma \vdash \vee_l(\pi) : \phi \vee \psi} \qquad (\vee E)\ \frac{\Gamma \vdash \pi : \phi \vee \psi \qquad \Gamma, \alpha : \phi \vdash \pi' : \vartheta \qquad \Gamma, \beta : \psi \vdash \pi'' : \vartheta}{\Gamma \vdash (\lambda \vee_l (\alpha).\pi' \,; \lambda \vee_r (\beta).\pi'')\ \pi : \vartheta}$$

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$$(\vee I_l)\ \frac{\Gamma \vdash \pi : \phi}{\Gamma \vdash \vee_l(\pi) : \phi \vee \psi} \qquad (\vee E)\ \frac{\Gamma \vdash \pi : \phi \vee \psi \qquad \Gamma, \alpha : \phi \vdash \pi' : \vartheta \qquad \Gamma, \beta : \psi \vdash \pi'' : \vartheta}{\Gamma \vdash (\lambda \vee_l (\alpha).\pi' \,; \lambda \vee_r (\beta).\pi'')\ \pi : \vartheta}$$

- The NDM formalization of higher-order logic gives the rules for higher-order quantifiers.

  Predicates defined by induction have to be studied in this view.