Strong normalization in the simply-typed system of the ρ-cube

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joint work with C. Kirchner, L. Liquori, H. Cirstea

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Untyped translation

Dealing with types

Strong normalization

What's next

Encoding the ρ-calculus in λ-calculus

$$[\![X]\!] \triangleq X$$

$$[\![f_i]\!] \triangleq \lambda x_1 \ldots \lambda x_{\alpha_i}.\ (\lambda z_1 \ldots \lambda z_S.(z_i\ x_1 \ldots x_{\alpha_i}))$$

$$[\![f_i {\bullet} B_1 \ldots {\bullet} B_{\alpha_i}]\!] \triangleq \lambda z_1 \ldots \lambda z_S.(z_i\ B_1 \ldots B_{\alpha_i})$$

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$$[\![f_i \bullet B_1 \ldots \bullet B_{\alpha_i}]\!] \triangleq \lambda z_1 \ldots \lambda z_S.(z_i\ B_1 \ldots B_{\alpha_i})$$

$$[\![(f_i \bullet X_1 \bullet \ldots \bullet X_p) \to A]\!] \triangleq \lambda y.(y\ \underbrace{x_\perp \ldots x_\perp}_{(\alpha_i - p)}\ \underbrace{x_\perp \ldots x_\perp}_{(i-1)}\ \lambda X_1 \ldots \lambda X_p.[\![A]\!]\ \underbrace{x_\perp \ldots x_\perp}_{(S-i)})$$

Encoding the ρ-calculus in λ-calculus

$$
\begin{aligned}
[\![X]\!] &\triangleq X \\
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[\![f_i{\bullet}B_1 \ldots {\bullet}B_{\alpha_i}]\!] &\triangleq \lambda z_1 \ldots \lambda z_S.(z_i\ B_1 \ldots B_{\alpha_i}) \\
[\![(f_i{\bullet}X_1{\bullet} \ldots {\bullet}X_p) \to A]\!] &\triangleq \lambda y.(y\ \underbrace{x_\perp \ldots x_\perp}_{(\alpha_i - p)}\ \underbrace{x_\perp \ldots x_\perp}_{(i-1)}\ \lambda X_1 \ldots \lambda X_p.[\![A]\!]\ \ \underbrace{x_\perp \ldots x_\perp}_{(S-i)}) \\
[\![X \to A]\!] &\triangleq \lambda X.[\![A]\!] \\
[\![A{\bullet}B]\!] &\triangleq [\![A]\!][\![B]\!]
\end{aligned}
$$

Encoding the ρ-calculus in λ-calculus

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[\![X \to A]\!] &\triangleq \lambda X.[\![A]\!] \\
[\![A \bullet B]\!] &\triangleq [\![A]\!][\![B]\!] \\
[\![A; B]\!] &\triangleq \lambda x_1 \ldots \lambda x_\alpha.\ \Big(\big(\lambda z.([\![A]\!] x_1 \ldots x_\alpha)\big)([\![B]\!] x_1 \ldots x_\alpha)\Big)
\end{aligned}
$$

Examples

$$(Success) \qquad \underbrace{\Big(\lambda y.(y(\lambda X.X)x_{\perp})\Big)}_{[\![f^{\bullet}X \to X]\!]}\underbrace{\lambda z_1 \lambda z_2.(z_1 Y)}_{[\![f^{\bullet}Y]\!]}$$

Examples

$$(Success) \qquad \underbrace{\Big(\lambda y.(y(\lambda X.X)x_\perp)\Big)}_{[\![f^\bullet X \to X]\!]} \underbrace{\lambda z_1 \lambda z_2.(z_1 Y)}_{[\![f^\bullet Y]\!]}$$

$$\mapsto_\beta \quad \underbrace{\Big(\lambda z_1 \lambda z_2.(z_1 Y)\Big)(\lambda X.X)x_\perp}_{[\![[f^\bullet X \ll f^\bullet Y]X]\!]}$$

Examples

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$$\mapsto_\beta \quad \big(\lambda z_2.((\lambda X.X)Y)\big)x_\perp$$

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Examples

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Examples

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$$(Failure) \qquad \underbrace{\big(\lambda y.(y(\lambda X.X)x_{\perp})\big)}_{[\![f^{\bullet}X\to X]\!]}\underbrace{\lambda z_1\lambda z_2.(z_2 Y)}_{[\![g^{\bullet}Y]\!]}$$

Examples

$$(Success) \quad \underbrace{\left(\lambda y.(y(\lambda X.X)x_\perp)\right)}_{[\![f^\bullet X \to X]\!]} \underbrace{\lambda z_1 \lambda z_2.(z_1 Y)}_{[\![f^\bullet Y]\!]}$$

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$$\mapsto_\beta \quad x_\perp Y$$

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Constants

Variables

Constraints

The type of a constant

$$\dfrac{\dfrac{\dfrac{\Gamma \vdash z_i : \sigma_1 \to \sigma_2 \to \beta \qquad (\forall n \in [1,2])\ \Gamma \vdash x_n : \sigma_n}{\Gamma \vdash z_i\ x_1\ x_2 : \beta}\ (\text{APP})}{x_\perp : \beta, \overrightarrow{x_n : \sigma_n} \vdash \lambda z_1 \ldots \lambda z_S.(z_i\ x_1\ x_2) : [\sigma_1 \to \sigma_2 \to \beta]^S \to \beta}\ (\text{ABS})}{x_\perp : \beta \vdash [\![f_i]\!] = \lambda x_1.\lambda x_2.\lambda z_1 \ldots \lambda z_S.(z_i\ x_1\ x_2) : \sigma_1 \to \sigma_2 \to [\sigma_1 \to \sigma_2 \to \beta]^S \to \beta}\ (\text{ABS}$$

where $[\sigma]^S \to \tau \quad \triangleq \quad \underbrace{\sigma \to \ldots \to \sigma}_{S} \to \tau$

$$\vdash [\![f_i(B_1, B_2)]\!] : [\sigma_1 \to \sigma_2 \to \beta]^S \to \beta$$

Type of $[\![f_i(X_1, X_2) \to A]\!]$

$$\dfrac{\vdash x_\perp : \sigma_1 \to \sigma_2 \to \tau \qquad \dfrac{X_1 : \sigma_1, X_2 : \sigma_2 \vdash [\![A]\!] : \tau}{\vdash \lambda X_1.\lambda X_2.[\![A]\!] : \sigma_1 \to \sigma_2 \to \tau}\ (\text{ABS})}{\dfrac{y : [\sigma_1 \to \sigma_2 \to \tau]^S \to \gamma \vdash y\ \overrightarrow{x_\perp}\ \left(\lambda X_1.\lambda X_2.[\![A]\!]\right)\ \overrightarrow{x_\perp} : \gamma}{\vdash \lambda y.\left(y\ \overrightarrow{x_\perp}\ \left(\lambda X_1.\lambda X_2.[\![A]\!]\right)\ \overrightarrow{x_\perp}\right) : \left([\sigma_1 \to \sigma_2 \to \tau]^S \to \gamma\right) \to \gamma}\ (\text{ABS})}\ (\text{APP})$$

$$[\sigma_1 \to \sigma_2 \to \tau]^S \to \gamma \ = \ [\sigma_1 \to \sigma_2 \to \beta]^S \to \beta \ \text{ thus } \ \tau = \beta = \gamma$$

$$\begin{aligned} \{\sigma_1, \ldots, \sigma_\alpha\} &\triangleq \Pi(\beta : *).\left([\sigma_1 \to \ldots \sigma_\alpha \to \beta]^S \to \beta\right) \\ \{\emptyset\} &\triangleq \Pi(\beta : *).\left([\beta]^S \to \beta\right) \end{aligned}$$

Enhanced translation

$$\vdash x_\perp \quad : \quad \perp$$

$$[\![f_i]\!] \triangleq \lambda x_1.\lambda x_2.\lambda(\beta : *)\ (\lambda z_1 \ldots \lambda z_S.(z_i x_1 x_2))$$

$$: \sigma_1 \to \sigma_2 \to \{\sigma_1, \sigma_2\}$$

$$[\![f_i(X_1, X_2) \to_\Delta A]\!] \triangleq \lambda y.\big(y\ \tau\ \overrightarrow{(x_\perp \tau_0)}\ \ \lambda X_1.\lambda X_2.[\![A]\!]\ \overrightarrow{(x_\perp \tau_0)}\big)$$

$$: \{\sigma_1, \sigma_2\} \to \tau$$

where $[\![\Gamma]\!] \vdash_{\mathsf{F}\omega} [\![A]\!] : \tau$

and $[\![\Gamma]\!] \vdash_{\mathsf{F}\omega} \lambda X_1.\lambda X_2.[\![A]\!] : \tau_0$

Constants

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Constraints

The type of a variable

$$(\Xi : *), (X : Y \rightarrow_{Y:\Xi} \Xi) \vdash_\rho X : Y \rightarrow_{Y:\Xi} \Xi$$

$$(\Xi : *) \vdash_\rho Y \rightarrow_{Y:\Xi} Y : Y \rightarrow_{Y:\Xi} \Xi$$

$$(\Xi : *), (a : \Xi) \vdash_\rho Y \rightarrow_{Y:\Xi} a : Y \rightarrow_{Y:\Xi} \Xi$$

$$(\Xi : *), (f : Y \rightarrow_{Y:\Xi} \Xi) \vdash_\rho f : Y \rightarrow_{Y:\Xi} \Xi$$

The type of a variable

$$(\Xi : *), (X : Y \rightarrow_{Y:\Xi} \Xi) \vdash_\rho X \quad : Y \rightarrow_{Y:\Xi} \Xi$$

$$(\Xi : *) \vdash_\rho Y \rightarrow_{Y:\Xi} Y : Y \rightarrow_{Y:\Xi} \Xi$$

$$(\Xi : *), (a : \Xi) \vdash_\rho Y \rightarrow_{Y:\Xi} a : Y \rightarrow_{Y:\Xi} \Xi$$

$$(\Xi : *), (f : Y \rightarrow_{Y:\Xi} \Xi) \vdash_\rho f \quad : Y \rightarrow_{Y:\Xi} \Xi$$

$$\Gamma \vdash_{\mathsf{F}\omega} \lambda(\beta_Y : *).\lambda(Y : \beta_Y).Y \quad : \Pi(\beta_Y : *).\big(\beta_Y \rightarrow \beta_Y\big)$$

$$\Gamma \vdash_{\mathsf{F}\omega} \lambda(\beta_Y : *).\lambda(Y : \beta_Y).[\![a]\!] : \Pi(\beta_Y : *).\big(\beta_Y \rightarrow \{\emptyset\}\big)$$

$$\Gamma \vdash_{\mathsf{F}\omega} [\![f]\!] \quad : \Pi(\beta_Y : *).\big(\beta_Y \rightarrow \{\beta_Y\}\big)$$

Use of types depending on types

$$[\![Y \to_{Y:\Xi} \Xi]\!]^X = \Pi(\beta_Y : *).\big(\beta_Y \to \beta_X \beta_Y)$$

with $\beta_X : * \to *$

$Y \to Y$	$\Pi(\beta_Y : *).\beta_Y \to \beta_Y$	$\beta_X := \lambda(\beta : *).\beta$
$Y \to a$	$\Pi(\beta_Y : *).\beta_Y \to \{\emptyset\}$	$\beta_X := \lambda(\beta : *).\{\emptyset\}$
f	$\Pi(\beta_Y : *).\beta_Y \to \{\beta_Y\}$	$\beta_X := \lambda(\beta : *).\{\beta\}$

Generalization for arbitrary types and contexts

$$\mathbb{K}(Y_1 \rightarrow_{Y_1:\Phi_1} \dots Y_n \rightarrow_{Y_n:\Phi_n} \Xi) \triangleq \text{a suitable kind for } \beta_X$$

$$[\![Y_1 \rightarrow_{Y_1:\Phi_1} \dots Y_n \rightarrow_{Y_n:\Phi_n} \Xi]\!]^X \triangleq \Pi\beta_{Y_1} : \mathbb{K}(\Phi_{Y_1}).([\![\Phi_{Y_1}]\!]^{Y_1} \rightarrow \dots$$
$$\Pi\beta_{Y_n} : \mathbb{K}(\Phi_{Y_n}).([\![\Phi_{Y_n}]\!]^{Y_n} \rightarrow \beta_X \overrightarrow{\beta_{Y_i}}))$$

$$[\![\emptyset]\!] \triangleq x_\bot : \bot$$

$$[\![\Gamma, X : \Phi]\!] \triangleq [\![\Gamma]\!],\ \beta_X : \mathbb{K}(\Phi)_\emptyset,\ X : [\![\Phi]\!]^X_\emptyset$$

Constants

Variables

Constraints

The type of an application

Suppose $\Sigma, \Gamma \vdash_\rho A \bullet B : [f(X_1 \ldots X_n) \ll_\Delta B]\Psi$

and $[\![\Gamma]\!] \vdash_{\mathsf{F}\omega} [\![B]\!] : \sigma$

$$[\![A \bullet B]\!] \triangleq [\![A]\!]\ \tau_{X_1} \ldots \tau_{X_n}\ [\![B]\!]$$

$$\text{if} \qquad \sigma =_\beta \{[\![\Phi_1]\!]^{X_1} \ldots [\![\Phi_n]\!]^{X_n}\}[\beta_{X_1} := \tau_{X_1} \ldots \beta_X := \tau_X]$$

$$[\![A \bullet B]\!] \triangleq \lambda \overrightarrow{\beta'_X : \mathbb{K}(\Phi_X)}.\ \lambda \left(w : \sigma \to \left\{[\![\Phi_1]\!]^{X'_1} \ldots [\![\Phi_n]\!]^{X'_n}\right\}\right). \quad \left([\![A]\!]\ \overrightarrow{\beta'_X}\ (w[\![B]\!])\right)$$

$$\text{if} \ \nexists \overrightarrow{\tau_X},\ \sigma =_\beta \{[\![\Phi_1]\!]^{X_1} \ldots [\![\Phi_n]\!]^{X_n}\}[\overrightarrow{\beta_X := \tau_X}]$$

Constraint resolution

$$[\![A{\bullet}B]\!] \triangleq \lambda\overrightarrow{\beta'_X : \mathbb{K}(\Phi_X)}.\,\lambda\left(w : \sigma \to \left\{[\![\Phi_1]\!]^{X'_1} \ldots [\![\Phi_n]\!]^{X'_n}\right\}\right).\ \left([\![A]\!]\ \overrightarrow{\beta'_X}\ (w[\![B]\!])\right)$$

If a subsequent instantiation θ_0 of some free type variables (in σ) enforces:

$$\exists\overrightarrow{\tau_X},\ \sigma\,\theta_0 \quad \stackrel{\beta}{=} \quad \left\{[\![\Phi_1]\!]^{X'_1} \ldots [\![\Phi_n]\!]^{X'_n}\right\}[\overrightarrow{\beta_X := \tau_X}]$$

Constraint resolution

$$[\![A \bullet B]\!] \triangleq \lambda \overrightarrow{\beta'_X : \mathbb{K}(\Phi_X)}. \lambda \left(w : \sigma \to \left\{ [\![\Phi_1]\!]^{X'_1} \dots [\![\Phi_n]\!]^{X'_n} \right\} \right). \left([\![A]\!] \; \overrightarrow{\beta'_X} \; (w [\![B]\!]) \right)$$

If a subsequent instantiation θ_0 of some free type variables (in σ) enforces:

$$\exists \overrightarrow{\tau_X}, \; \sigma \, \theta_0 \quad \stackrel{\beta}{=} \; \left\{ [\![\Phi_1]\!]^{X'_1} \dots [\![\Phi_n]\!]^{X'_n} \right\} [\overrightarrow{\beta_X := \tau_X}]$$

- In the term, using an enhanced η-expansion, $\quad w := \lambda x.x$

  Before the constraint is resolved in the types, there can be no σ-reduction...

Constraint resolution

$$[\![A{\bullet}B]\!] \triangleq \lambda\overrightarrow{\beta'_X : \mathbb{K}(\Phi_X)}.\, \lambda\left(w : \sigma \to \left\{[\![\Phi_1]\!]^{X'_1} \ldots [\![\Phi_n]\!]^{X'_n}\right\}\right).\ \left([\![A]\!]\ \overrightarrow{\beta'_X}\ (w[\![B]\!])\right)$$

If a subsequent instantiation θ_0 of some free type variables (in σ) enforces:

$$\exists\overrightarrow{\tau_X},\ \sigma\,\theta_0 \quad \stackrel{\beta}{=} \ \left\{[\![\Phi_1]\!]^{X'_1} \ldots [\![\Phi_n]\!]^{X'_n}\right\}[\overrightarrow{\beta_X := \tau_X}]$$

- In the term, using an enhanced η-expansion, $\quad w := \lambda x.x$

  Before the constraint is resolved in the types, there can be no σ-reduction...

- In the types, nothing to be done !

$$\vdash_{\mathsf{F}\omega} w : \sigma\,\theta_0 \to \left\{[\![\Phi_1]\!]^{X'_1} \ldots [\![\Phi_n]\!]^{X'_n}\right\}[\overrightarrow{\beta_X := \tau_X}]$$

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Steps of the proof

- Faithful reductions

- Typability of the translated terms

- Strong normalization

Steps of the proof

- Faithful reductions

  If $A \mapsto_{\rho\sigma\delta} B$, then $[\![A]\!] \overset{\beta}{\mapsto\!\!\!\!\twoheadrightarrow} [\![B]\!]$ in at least one step.

- Typability of the translated terms

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Steps of the proof

- Faithful reductions

  If $A \mapsto_{\rho\sigma\delta} B$, then $[\![A]\!] \stackrel{\beta}{\mapsto\!\!\!\twoheadrightarrow} [\![B]\!]$ in at least one step.

- Typability of the translated terms

  Giving a fresh variable Z,

$$\Sigma, \Gamma \vdash_\rho A : \Phi \quad \Rightarrow \quad \exists \tau_A, \quad [\![\Gamma]\!] \vdash_{\mathsf{F}\omega} [\![A]\!] : [\![\Phi]\!]^Z_\emptyset [\beta_Z := \tau_A]$$

- Strong normalization

Steps of the proof

- Faithful reductions

  If $A \mapsto_{\rho\sigma\delta} B$, then $[\![A]\!] \overset{\beta}{\mapsto\!\!\!\rightarrow} [\![B]\!]$ in at least one step.

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  Giving a fresh variable Z,

$$\Sigma, \Gamma \vdash_\rho A : \Phi \quad \Rightarrow \quad \exists \tau_A, \quad [\![\Gamma]\!] \vdash_{\mathsf{F}\omega} [\![A]\!] : [\![\Phi]\!]^Z_\emptyset [\beta_Z := \tau_A]$$

- Strong normalization

  If $\Sigma, \Gamma \vdash_\rho A : \Phi$ then A and Φ are strongly normalizing.

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What's next

Towards an extension of Curry-Howard

- Some open questions
  - ➪ What do the types $A \to C$ and $[A \ll B].C$ mean ?
  - ➪ Are the structures suitable for products or sums ?

Towards an extension of Curry-Howard

- Some open questions

➪ What do the types $A \to C$ and $[A \ll B].C$ mean ?

➪ Are the structures suitable for products or sums ?

- Some partial answers

➪ The λ-term $[\![f(X_1 \ldots X_n)]\!]$ has type $\bigwedge_{i=1..n} X_i$.

➪ Proof assistant based on the ρ-calculus: deduction and computation at the same level.

➪ Use of the typed fixpoints to model trustworthy computations steps.

Example with a delayed constraint

$$(Y \to (f \bullet X \to X) \bullet Y) \bullet (f \bullet a)$$

Example with a delayed constraint

$$
\begin{aligned}
& (Y \to (f \bullet X \to X) \bullet Y) \bullet (f \bullet a) \\
\mapsto_\rho \quad & (Y \to [f \bullet X \ll Y] X) \bullet (f \bullet a)
\end{aligned}
$$

Example with a delayed constraint

$$
\begin{aligned}
& (Y \to (f \bullet X \to X) \bullet Y) \bullet (f \bullet a) \\
\mapsto_\rho \quad & (Y \to [f \bullet X \ll Y] X) \bullet (f \bullet a) \\
\mapsto_\rho \quad & [Y \ll f \bullet a][f \bullet X \ll Y] X
\end{aligned}
$$

Example with a delayed constraint

$$
\begin{array}{ll}
 & (Y \rightarrow (f{\bullet}X \rightarrow X){\bullet}Y){\bullet}(f{\bullet}a) \\
\mapsto_\rho & (Y \rightarrow [f{\bullet}X \ll Y]X){\bullet}(f{\bullet}a) \\
\mapsto_\rho & [Y \ll f{\bullet}a][f{\bullet}X \ll Y]X \\
\mapsto_\sigma & [f{\bullet}X \ll f{\bullet}a]X
\end{array}
$$

Example with a delayed constraint

$$
\begin{array}{ll}
 & (Y \to (f\bullet X \to X)\bullet Y)\bullet(f\bullet a) \\
\mapsto_\rho & (Y \to [f\bullet X \ll Y]X)\bullet(f\bullet a) \\
\mapsto_\rho & [Y \ll f\bullet a][f\bullet X \ll Y]X \\
\mapsto_\sigma & [f\bullet X \ll f\bullet a]X \\
\mapsto_\sigma & a
\end{array}
$$

Example with a delayed constraint (cont'd)

$$(\lambda Y.\Big(\underbrace{(\lambda y.(y x_{\perp}(\lambda X.X)))}_{[\![f^{\bullet} X \to X]\!]}\, Y\Big))\Big(\underbrace{(\lambda x_1.\lambda z_1 \lambda z_2.(z_2 x_1))}_{[\![f]\!]}\underbrace{(\lambda u_1 \lambda u_2.u_1)}_{[\![a]\!]}\Big)$$

Example with a delayed constraint (cont'd)

$$(\lambda Y.\Big(\underbrace{(\lambda y.(y x_{\perp}(\lambda X.X)))}_{[\![f^{\bullet} X \to X]\!]} Y\Big))\Big(\underbrace{(\lambda x_1.\lambda z_1 \lambda z_2.(z_2 x_1))}_{[\![f]\!]}\underbrace{(\lambda u_1 \lambda u_2.u_1)}_{[\![a]\!]}\Big)$$

$$\longmapsto_{\beta} \quad \underbrace{(\lambda Y.(Y x_{\perp}(\lambda X.X)))}_{[\![Y \to [f^{\bullet} X \ll Y] X]\!]}\Big(\underbrace{(\lambda x_1.\lambda z_1 \lambda z_2.(z_2 x_1))}_{[\![f]\!]}\underbrace{(\lambda u_1 \lambda u_2.u_1)}_{[\![a]\!]}\Big)$$

Example with a delayed constraint (cont'd)

$$\begin{aligned}
& (\lambda Y.\Big(\underbrace{(\lambda y.(y x_\perp(\lambda X.X)))}_{[\![f^\bullet X\to X]\!]}\,Y\Big))\Big(\underbrace{(\lambda x_1.\lambda z_1\lambda z_2.(z_2 x_1))}_{[\![f]\!]}\,\underbrace{(\lambda u_1\lambda u_2.u_1)}_{[\![a]\!]}\Big) \\
\mapsto_\beta\quad & \underbrace{\big(\lambda Y.(Y x_\perp(\lambda X.X))\big)}_{[\![Y\to[f^\bullet X\ll Y]X]\!]}\Big(\underbrace{(\lambda x_1.\lambda z_1\lambda z_2.(z_2 x_1))}_{[\![f]\!]}\,\underbrace{(\lambda u_1\lambda u_2.u_1)}_{[\![a]\!]}\Big) \\
\mapsto_\beta\quad & \big(\lambda Y.(Y x_\perp(\lambda X.X))\big)\big(\lambda z_1\lambda z_2.(z_2\,\underbrace{(\lambda u_1\lambda u_2.u_1)}_{[\![a]\!]})\big)
\end{aligned}$$

Example with a delayed constraint (cont'd)

$$(\lambda Y.\Big(\underbrace{(\lambda y.(y x_\perp(\lambda X.X)))}_{[\![f^\bullet X\to X]\!]}\,Y\Big))\Big(\underbrace{(\lambda x_1.\lambda z_1\lambda z_2.(z_2 x_1))}_{[\![f]\!]}\,\underbrace{(\lambda u_1\lambda u_2.u_1)}_{[\![a]\!]}\Big)$$

$$\mapsto_\beta \quad \underbrace{\big(\lambda Y.(Y x_\perp(\lambda X.X))\big)}_{[\![Y\to[f^\bullet X\ll Y]X]\!]}\Big(\underbrace{(\lambda x_1.\lambda z_1\lambda z_2.(z_2 x_1))}_{[\![f]\!]}\,\underbrace{(\lambda u_1\lambda u_2.u_1)}_{[\![a]\!]}\Big)$$

$$\mapsto_\beta \quad \big(\lambda Y.(Y x_\perp(\lambda X.X))\big)\big(\lambda z_1\lambda z_2.(z_2\,\underbrace{(\lambda u_1\lambda u_2.u_1)}_{[\![a]\!]})\big)$$

$$\mapsto_\beta \quad \big(\lambda z_1\lambda z_2.(z_2(\lambda u_1\lambda u_2.u_1))\big)x_\perp(\lambda X.X) \qquad (f\neq a)$$

Example with a delayed constraint (cont'd)

$$
\begin{aligned}
& (\lambda Y.\Big(\underbrace{(\lambda y.(y x_\perp (\lambda X.X)))}_{[\![f^\bullet X \to X]\!]} Y \Big)) \Big(\underbrace{(\lambda x_1.\lambda z_1 \lambda z_2.(z_2 x_1))}_{[\![f]\!]} \underbrace{(\lambda u_1 \lambda u_2.u_1)}_{[\![a]\!]} \Big) \\
\mapsto_\beta \quad & \underbrace{\big(\lambda Y.(Y x_\perp (\lambda X.X))\big)}_{[\![Y \to [f^\bullet X \ll Y] X]\!]} \Big(\underbrace{(\lambda x_1.\lambda z_1 \lambda z_2.(z_2 x_1))}_{[\![f]\!]} \underbrace{(\lambda u_1 \lambda u_2.u_1)}_{[\![a]\!]} \Big) \\
\mapsto_\beta \quad & \big(\lambda Y.(Y x_\perp (\lambda X.X))\big) \big(\lambda z_1 \lambda z_2.(z_2 \underbrace{(\lambda u_1 \lambda u_2.u_1)}_{[\![a]\!]})\big) \\
\mapsto_\beta \quad & \big(\lambda z_1 \lambda z_2.(z_2 (\lambda u_1 \lambda u_2.u_1))\big) x_\perp (\lambda X.X) \qquad\qquad (f \neq a) \\
\mapsto_\beta \quad & \big(\lambda z_2.(z_2 (\lambda u_1 \lambda u_2.u_1))\big) (\lambda X.X) \qquad\qquad (f = f)
\end{aligned}
$$

Example with a delayed constraint (cont'd)

$$(\lambda Y.\Big(\underbrace{(\lambda y.(y x_\perp(\lambda X.X)))}_{[\![f^\bullet X \to X]\!]} Y\Big))\Big(\underbrace{(\lambda x_1.\lambda z_1 \lambda z_2.(z_2 x_1))}_{[\![f]\!]}\underbrace{(\lambda u_1 \lambda u_2.u_1)}_{[\![a]\!]}\Big)$$

$$\mapsto_\beta \quad \underbrace{\big(\lambda Y.(Y x_\perp(\lambda X.X))\big)}_{[\![Y \to [f^\bullet X \ll Y] X]\!]}\Big(\underbrace{(\lambda x_1.\lambda z_1 \lambda z_2.(z_2 x_1))}_{[\![f]\!]}\underbrace{(\lambda u_1 \lambda u_2.u_1)}_{[\![a]\!]}\Big)$$

$$\mapsto_\beta \quad \big(\lambda Y.(Y x_\perp(\lambda X.X))\big)\big(\lambda z_1 \lambda z_2.(z_2 \underbrace{(\lambda u_1 \lambda u_2.u_1)}_{[\![a]\!]})\big)$$

$$\mapsto_\beta \quad \big(\lambda z_1 \lambda z_2.(z_2(\lambda u_1 \lambda u_2.u_1))\big) x_\perp (\lambda X.X) \qquad (f \neq a)$$

$$\mapsto_\beta \quad \big(\lambda z_2.(z_2(\lambda u_1 \lambda u_2.u_1))\big)(\lambda X.X) \qquad (f = f)$$

$$\mapsto_\beta \quad (\lambda X.X)(\lambda u_1 \lambda u_2.u_1)$$

Example with a delayed constraint (cont'd)

$$
\begin{array}{ll}
 & (\lambda Y.\Big(\underbrace{(\lambda y.(y x_\perp(\lambda X.X)))}_{[\![f^\bullet X\to X]\!]} Y\Big))\Big(\underbrace{(\lambda x_1.\lambda z_1\lambda z_2.(z_2 x_1))}_{[\![f]\!]}\underbrace{(\lambda u_1\lambda u_2.u_1)}_{[\![a]\!]}\Big) \\
\mapsto_\beta & \underbrace{\big(\lambda Y.(Y x_\perp(\lambda X.X))\big)}_{[\![Y\to[f^\bullet X\ll Y]X]\!]}\Big(\underbrace{(\lambda x_1.\lambda z_1\lambda z_2.(z_2 x_1))}_{[\![f]\!]}\underbrace{(\lambda u_1\lambda u_2.u_1)}_{[\![a]\!]}\Big) \\
\mapsto_\beta & \big(\lambda Y.(Y x_\perp(\lambda X.X))\big)\big(\lambda z_1\lambda z_2.(z_2\underbrace{(\lambda u_1\lambda u_2.u_1)}_{[\![a]\!]})\big) \\
\mapsto_\beta & \big(\lambda z_1\lambda z_2.(z_2(\lambda u_1\lambda u_2.u_1))\big)x_\perp(\lambda X.X) \qquad\qquad (f\neq a) \\
\mapsto_\beta & \big(\lambda z_2.(z_2(\lambda u_1\lambda u_2.u_1))\big)(\lambda X.X) \qquad\qquad (f=f) \\
\mapsto_\beta & (\lambda X.X)(\lambda u_1\lambda u_2.u_1) \\
\mapsto_\beta & (\lambda u_1\lambda u_2.u_1) \\
= & [\![a]\!]
\end{array}
$$