Combining λ-calculus and rewriting: brief history and open problems

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Problem

confluence (CR) and strong normalization (SN) of $\beta \cup \mathcal{R}_1 \cup \mathcal{R}_\omega$?

$\mathcal{R}_1$: usual first-order rewriting (algebraic LHS/RHS, $\mathcal{R}_1 \cap \mathcal{F}_\omega = \emptyset$)

$\mathcal{R}_\omega$: higher-order rewriting, 2 cases:

– LHS's are algebraic: syntactic/first-order matching is sufficient

– LHS's are higher-order patterns (Klop, 1980; Miller, 1989)

<u>emark:</u> $\beta \cup \mathcal{R}$ SN and $\mathcal{R}$ locally CR $\Rightarrow$ $\beta \cup \mathcal{R}$ CR (Newman's Lemma)

Three research directions

- $\beta \in \mathcal{R}_1$: β is encoded by first-order rewrite rules
- $\beta \in \mathcal{R}_\omega$: β isn't distinguished from the other higher-order rewrite rules
- $\beta \notin \mathcal{R}$: β receives a particular treatment

mark: I will not speak about pattern-calculi... :-([Julien?]

$\beta \subseteq \mathcal{R}_1$

- 1972: de Bruijn: first-order representation of λ-terms
- 1991: Abadi, Cardelli, Curien and Lévy: first-order representation of β
- . . . : many works!
- 2000: Bonelli, Kesner and Ríos: first-order representation of higher-order rewrite rules ($\mathcal{R}_\omega \subseteq \mathcal{R}_1$)

$\boxed{\beta \in \mathcal{R}_\omega}$

mark: these results apply on the other cases too !

- 1980: Klop: Combinatory Reduction Systems (CRS)
  - (left-linear and no critical pair) orthogonal $\Rightarrow$ CR
  - $\exists\ \mathcal{R}_1$ CR and non left-linear s.t. $\beta \cup \mathcal{R}_1$ not CR (since β not SN)
- 1990: Khasidashvili: Expression Reduction Systems (ERS)
  a variant/extension of CRS ($\perp \Rightarrow$ CR)
- 1991: Nipkow: Higher-order Rewrite Systems (HRS)
  rewriting modulo $\beta\eta$ in $\lambda^{\rightarrow}$-calculus ($\perp \Rightarrow$ CR, Critical Pair Lemma)
- 1993: Van de Pol: termination of HRS's by (higher-order) interpretations

- 1994: v.Oostrom and v.Raamsdonk: Higher-Order Rewriting Systems (HORS)
rewriting = replacement/grafting modulo a substitution calculus $\mathcal{SC}$ ($\perp \Rightarrow$ CR)
examples: TRS, CRS, HRS, ERS ($\mathcal{SC} = \beta\overline{\eta}$)

- 1995: Van Oostrom: left-linear with development closed critical pairs $\Rightarrow$ CR

- 1999: Van Oostrom: (left-linear and trivial critical pairs) weakly orthogonal $\Rightarrow$ WN (outermost-fair strategies)

- 2000: Blanqui: $\mathcal{R}_1$ SN and non-duplicating and $\mathcal{R}_\omega$ structurally decreasing simply-typed HRS/CRS $\Rightarrow$ $\mathcal{R}_1 \cup \mathcal{R}_\omega$ SN

on-duplicating: no variable occurs more in the RHS than in the LHS
ructurally decreasing: in RHS, recursive calls are done on arguments structurally
naller than the arguments of the LHS

$\beta \notin \mathcal{R}$ when typing doesn't depend on rewriting

- 1988: Breazu-Tannen: $\mathcal{R}_1$ CR $\Rightarrow \lambda^{\rightarrow} \cup \mathcal{R}_1$ CR
- 1989: Breazu-Tannen, Gallier and Okada: $\mathcal{R}_1$ SN $\Rightarrow \lambda^{\rightarrow} \cup \mathcal{R}_1$ SN
- 1990: Barbanera: $\mathcal{R}_1$ SN $\Rightarrow$ CC $\cup \mathcal{R}_1$ SN
- 1991: Dougherty: β SN and $\mathcal{R}_1$ CR/SN $\Rightarrow \beta \cup \mathcal{R}_1$ CR/SN
- 1992: Müller: $\mathcal{R}_1$ CR and left-linear $\Rightarrow \beta \cup \mathcal{R}_1$ CR (subsumed by Van Oostrom)

ith algebraic higher-order rewriting:

- 1991: Jouannaud and Okada: $\mathcal{R}_1$ SN and non-duplicating and $\mathcal{R}_\omega^{\rightarrow}$ structurally decreasing $\Rightarrow \lambda^{\rightarrow} \cup \mathcal{R}$ SN
- 1999: Jouannaud and Rubio: $\mathcal{R}$ in $\lambda^{\rightarrow}$-HORPO $\Rightarrow \lambda^{\rightarrow} \cup \mathcal{R}$ SN

About non-duplication

- 1988: Toyama: $\mathcal{R}_1$ and $\mathcal{R}_2$ SN $\not\Rightarrow$ $\mathcal{R}_1 \uplus \mathcal{R}_2$ SN
- 1988: Rusinowitch: $\mathcal{R}_1$ and $\mathcal{R}_2$ SN and non-duplicating $\Rightarrow$ $\mathcal{R}_1 \uplus \mathcal{R}_2$ SN
- 1992: Gramlich: $\mathcal{R}_1$ and $\mathcal{R}_2$ SN and $\mathcal{R}_1 \uplus \mathcal{R}_2$ not SN $\Rightarrow$ $\mathcal{R}_1$ non-duplicating and $\mathcal{R}_2$ collapsing (RHS = variable) (or vice-versa)
- 1995: Melliès: $\lambda\sigma$ is not SN (several duplicating and collapsing rules)
- 1995: Kurihara and Ohuchi: $\mathcal{R}_1$ and $\mathcal{R}_2$ SN $\Rightarrow$ $\mathcal{R}_1 \uplus \mathcal{R}_2$ WN with sharing

$\beta \notin \mathcal{R}$ when typing depends on rewriting

(with algebraic higher-order rewriting)

$$(\text{conv}) \quad \frac{\Gamma \vdash t : T \quad T \downarrow_{\beta\mathcal{R}} T'}{\Gamma \vdash t : T'}$$

- 1993: Barbanera, Fernández and Geuvers: $\mathcal{R}_1$ SN and object-level $\mathcal{R}_\omega^{\rightarrow}$ structurally decreasing $\Rightarrow$ CC $\cup \mathcal{R}$ SN
- 2000: Walukiewicz: $\mathcal{R}$ in CC-HORPO $\Rightarrow$ CIC $\cup \mathcal{R}$ SN

remark: all previous works only consider object-level rewrite rules

with type-level rewrite rules:

- 2001: Blanqui: $\mathcal{R}_1$ SN and non-duplicating, $\mathcal{R}_\omega$ structurally decreasing, no type-level critical pair and $\beta \cup \mathcal{R}$ CR $\Rightarrow$ CC $\cup \mathcal{R}$ SN
- 2003: Blanqui: idem with rewriting modulo object-level AC-like theories
- 2004: Blanqui: idem with $\mathcal{R}_\omega$ size decreasing ($e.g.$ $f(sx)y \rightarrow s(f(-xy)y)$)

mark: see the prototype of Coq+rewriting on the Coq CVS server !
ag `recriture`)

Open problems

- having type-level critical pairs ?
- $\mathcal{R}$ locally CR, $\mathcal{R}_1$ SN (and non-dup), $\mathcal{R}_\omega$ size decreasing
  $\Rightarrow$ CC $\cup\,\mathcal{R}$ SN (with sharing) and CR ?
- having conditional rewriting ?
- efficient procedure for testing $\downarrow_{\beta\mathcal{R}}$?
- extending modularity results for first-order systems to higher-order systems ?

Application/link to ρ-calculus

- ρ-calculus = HORS with particular substitution calculus $\mathcal{SC}$?
- ρ-calculus with higher-order pattern matching *à la* Klop/Miller ?
- application of the positivity-based SN technique to ρCC ?